

*Rapid Note***Fractal dimension of Siegel disc boundaries**A.D. Burbanks^a, A.H. Osbaldestin^b, and A. Stirnemann

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Abstract. Using renormalization techniques, we provide rigorous computer-assisted bounds on the Hausdorff dimension of the boundary of Siegel discs. Specifically, for Siegel discs with golden mean rotation number and quadratic critical points we show that the Hausdorff dimension is less than 1.08523. This is done by exploiting a previously found renormalization fixed point and expressing the Siegel disc boundary as the attractor of an associated Iterated Function System.

PACS. 05.45.+b Theory and models of chaotic systems – 47.53.+n Fractals

The celebrated Kolmogorov-Arnold-Moser (KAM) theorem (see for instance [1]) is arguably one of the most important results in the theory of dynamical systems. For an integrable Hamiltonian system with N degrees of freedom, the motion in the $2N$ -dimensional phase space is confined to surfaces that are (topologically) N -dimensional tori. In brief, the KAM Theorem says that under a small enough perturbations from integrable, almost all of the motion remains restricted to surfaces that are topologically equivalent to N -dimensional tori. These tori support quasiperiodic motion, and persist up to some critical perturbation strength dependent on the rotation number. The analogous phenomenon for maps of the circle is important in the study of dissipative systems [2].

A precursor of the KAM theorem was the proof by Siegel [20] of the existence of a domain of linearizability, known as a *Siegel disc*, around an irrationally indifferent fixed point of a complex map. This problem forms a model for more complicated problems in nonlinear dynamics, in which transitions from quasiperiodicity to chaos are found to have universal characteristics. The mappings we have in mind are of the form

$$z \mapsto f(z) = \lambda z + O(z^2) \quad \text{with} \quad \lambda = \exp(2\pi i\nu).$$

Figure 1 shows a Siegel disc for a quadratic map. Analytic curves persist up to the fractal boundary.

In the above problems the crucial question is the existence of a conjugacy to rigid rotation. This problem is plagued by *small divisors*; it is heavily dependent on the number-theoretic properties of the rotation number ν . The

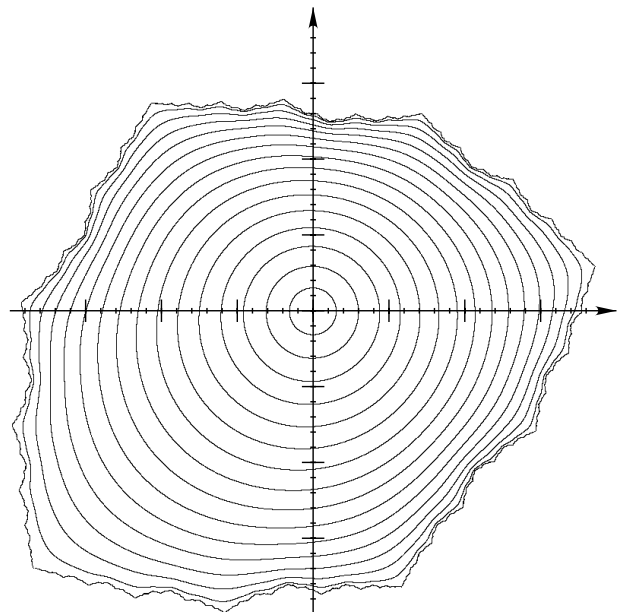


Fig. 1. Siegel disc for a golden mean quadratic mapping.

key breakthrough was the demonstration that a conjugacy exists provided the rotation number satisfies a *Diophantine condition*:

$$\left| \nu - \frac{p}{q} \right| > \frac{c}{|q|^{2+\mu}} \quad \text{for all integers } p, q,$$

for some constants $c > 0$, $\mu \geq 0$. Indeed for the Siegel disc, Yoccoz [24] has shown (for quadratic maps) that conjugacy exists if and only if the rotation number satisfies the

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weaker *Brjuno condition*:

$$\sum \frac{\log q_{n+1}}{q_n} < +\infty,$$

where (q_n) are the denominators in the convergents of the continued fraction expansion of ν .

The transitions from quasiperiodicity to chaos in these problems, and the invariant structures present at a critical parameter value, are observed to possess universal scaling properties, which have been analysed using renormalization group techniques. Because of its simple continued fraction expansion, the simplest case is when the rotation number is the golden mean, $(\sqrt{5}-1)/2 = 0.61803\dots$, and most attention has focussed on this case. We too confine our attention to the golden mean.

Perhaps the most straightforward application of renormalization group methods to the study of universality in dynamical systems was the explanation offered by Feigenbaum [12] for the universal features observed in the period-doubling route to chaos in iterated unimodal maps of the interval. This explanation centres on the existence of a hyperbolic fixed point for a renormalization operator acting on a space of functions. Existence of the fixed point, and the nature of the spectrum of the derivative of the renormalization operator there, allows the deduction of universal features for all maps attracted to the fixed point under iteration of the operator.

A renormalization explanation for universality in circle maps was constructed by Ostlund *et al.* [19] and Feigenbaum *et al.* [13], and for area-preserving twist maps by MacKay [16]. These explanations rely on the existence of both a *simple* fixed point for the renormalization operator, responsible for the presence of quasiperiodic motion in the sub-critical case, and also a *critical* fixed point for the operator, that controls the universal scaling features associated with the critical parameter value in these problems.

Manton and Nauenberg [15] made corresponding numerical observations of universality for Siegel discs in complex maps, and a renormalization explanation of the type mentioned above was later offered by Widom [23].

There is currently a rich mixture of both analytical and computer-assisted rigorous results as well as much numerical and heuristic work. The existence and hyperbolicity of a fixed point for the Feigenbaum renormalization operator was established by Lanford [14], and Campanino *et al.* [5, 6] using rigorous computer-assisted means, and later by Epstein [10] by analytical methods. Analogous proofs for the case of period-doubling in area-preserving maps were given by Eckmann *et al.* [8]. A rigorous computer-assisted proof of the existence and hyperbolicity of the golden-mean critical circle maps fixed point, analogous to that of Lanford, was performed by Mestel [17], and Eckmann and Epstein have since provided an analytic proof [7]. Stirnemann [21] has given a computer-assisted proof of existence in the case of golden-mean Siegel discs.

There is still no analytic proof of the existence of a critical fixed point for Siegel discs, and no proof at all of the existence of a critical fixed point for twist maps.

A by-product of the computer-assisted proofs is that they also provide rigorous bounds on the universal fixed-point functions. We have been able to take these bounds and cast the Siegel disc boundary as the attractor of an associated iterated function system (IFS). We are then able to use results from the theory of IFSs to obtain a rigorous upper bound on the Hausdorff dimension of the boundary.

The result is that we are able to prove that the Hausdorff dimension is less than 1.08523.

A nonrigorous version of our method produces the bounds 1.00119 to 1.07967. A previous numerical estimate 1.01 appears in [18].

We use a different formulation of the operator from Widom (see [21]), and write the renormalization operator as the map of pairs:

$$(E(z), F(z)) \mapsto (C\alpha^{-1}F(\alpha Cz), C\alpha^{-1}FE(\alpha Cz)),$$

where $\alpha = F(0)$ to preserve the normalisation $E(0) = 1$, and where C means complex conjugate. The difference in our formulation is that α is a *complex* scaling factor. We study quadratic critical points, so the functions E and F are even and may thus be written in the form $E(z) = U(z^2)$ and $F(z) = V(z^2)$. The renormalization operator is then

$$(U(z), V(z)) \mapsto (C\alpha^{-1}V(\alpha^2 Cz), C\alpha^{-1}VQU(\alpha^2 Cz)),$$

with $\alpha = V(0)$, and where Q is the map $z \mapsto z^2$.

Stirnemann [21] proves the existence of a fixed-point pair, (U, V) , of this operator defined on domains D_U, D_V respectively. In [3], the existence of the (universal) invariant curve for the fixed-point pair was deduced by applying the *necklace construction* of [22], in which a sequence of sets (*domain pairs*) is constructed iteratively by applying the maps of the renormalization fixed-point pair to their domains. Under certain hypotheses, the domains converge to a piece of the invariant curve (the *necklace*). These hypotheses were verified in [3].

The mapping defining the necklace construction is

$$(M, N) \mapsto (\alpha^2 CN, (\alpha^2 CM) \cup (QU\alpha^2 CN)).$$

We may eliminate M from the corresponding fixed-point equation to obtain the mapping

$$N \mapsto (|\alpha|^4 N) \cup (QU\alpha^2 CN).$$

This second form is recognisable as a standard IFS (see for instance [11]) consisting of the two maps $|\alpha|^4$ and $QU\alpha^2 C$ acting on an initial (single) domain. By computer-assisted means we have verified that these maps are contractions. (Note that one of the maps, $|\alpha|^4$, is a similarity, whilst the other is an analytic function of $\bar{z} = Cz$.) The fact that the maps are contractions guarantees the existence of an IFS attractor. However, by bounding the contraction factors of the maps we may also deduce bounds on the Hausdorff dimension of the attractor. We use the result (see [11]) that if an IFS consists of n contractions with contraction

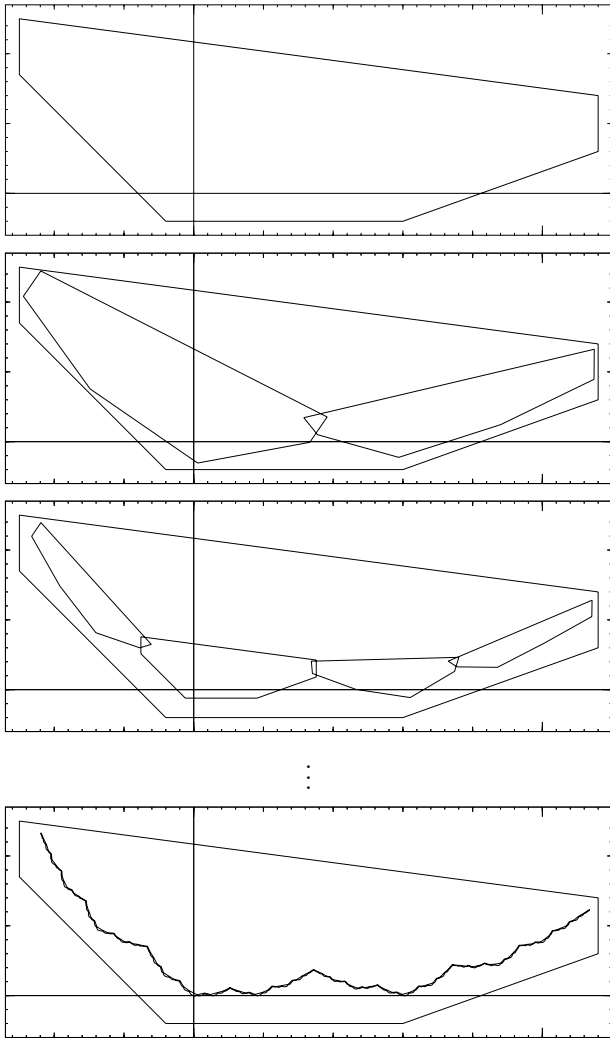


Fig. 2. Steps 0, 1, 2, and 7 in the application of the IFS.

factors κ_i then the solution s of the partition function equation

$$\sum_{i=1}^n \kappa_i^s = 1$$

is an upper bound on the Hausdorff dimension of the attractor of the IFS. Bounds on the contraction factors themselves are obtained by rigorously bounding the derivatives of the maps on convex domains and applying the mean value theorem. The sharpness of the bounds on the derivatives of these maps will dictate the sharpness of our eventual bounds on the Hausdorff dimension obtained by solving the partition function equation. In fact we consider iterates of the initial IFS to sharpen our bounds by reducing the domains on which the derivatives need to be calculated. This is at the expense of having to consider composites of the mappings.

Figure 2 shows iterates of the initial domain under the IFS converging on the attractor. Note the similarity with the boundary curve in Figure 1.

This approach will also yield lower bounds for the Hausdorff dimension. The only substantial extra condition to verify is the “open set condition” of [11] (Sect. 9.2). Unfortunately the only lower bounds our method has provided so far are less than one.

The dimension bounds calculated are for the universal fixed point of the renormalization transformation. Hausdorff dimension is invariant under this transformation, and so is shared by maps attracted to the fixed point.

Full details of these calculations appear in [4].

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